OPTIMUM MOTION OF A VARIABLE-MASS POINT WITH AERODYNAMIC FORCES IN THE UNIFORM CENTRAL GRAVITATIONAL FIELD

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The regimes of optimum motion of the variable-mass point in uniform central field, introduced into the analysis by G. E. Kuzmak, V. K. Isaev, and B. X. Davidson, are considered on the basis of the maximum principle [1]. The features introduced into the optimum programs of the jet force magnitude and direction control due to the aerodynamic forces are considered.

Let us consider the orbital motion of the variable-mass point in the Cartesian coordinate system OXY, where the point O coincides with the centre of gravity, taking into account the aerodynamic forces effects (the drag $Q = C_x qS$ and the lift $Y = C_y qS$, where $q = \rho V^2/2$ is the dynamic pressure).

The problem is treated under the following assumptions:

1. The trajectory of the motion is a quasicircular orbit of the radius R_{av};
2. The drag Q is of opposite direction to the absolute velocity vector $\vec{V}$, the lift $\vec{Y}$ is normal to the velocity vector $\vec{V}$ (Fig. 1);

3. The control is accomplished by the variation of the jet force magnitude and orientation $\vec{P}$;

4. The aerodynamic acceleration components $g(R_{av})n_x = Cx\ q^s/M$ and $g(R_{av})n_y = C_y\ qS/M$ are the linear functions of V:

$$g(R_{av}) = n_z B_z V; \qquad g(R_{av})n_y = B_y V_1 \qquad (1)$$

where

$$B_x = \frac{C_x \rho V S}{2M} = \text{const.}, \qquad B_y = \frac{C_y \rho V S}{2M}$$

The assumption (1) permits the gravitational field to be considered as a uniform central field, and the projections of the gravitational acceleration to be written in the form [2]:

$$g_x = -\nu^2 x, \qquad g_y = -\nu^2 y; \qquad \nu^2 = \frac{g(R_{av})}{R_{av}} = \text{const.}$$

Under this assumption the relation (1) may in particular suggest that there is considered the problem of the correction (or maintaining) circular orbit of the vehicle having constant aerodynamic characteristics C_x and C_y, total fuel consumption for the correcting maneuver being low. In effect for the motion along the quasicircular orbit we have

$$r = \sqrt{x^2 + y^2} = R_{av} + \Delta r, \qquad \Delta r \ll R_{av}$$

$$\rho = \rho(z) \cong \rho(R_{av}) = \rho^0;$$

$$V = \sqrt{u^2 + v^2} = V_{cir} + \Delta V, \qquad \Delta V \ll V_{cir}$$

where V_{cir} is the circular speed (or the velocity of the satellite moving along the orbit of the radius $r = R_{av}$). Assuming $C_x = C_x^0 = \text{const.}$, $C_y = C_y^0 = \text{const.}$, $M(t) = M(0) - \Delta M$, $\Delta M \ll M(0)$, from the above mentioned follows:

$$\frac{C_x^0 S(\rho - \Delta\rho)(V_{cir} - \Delta V)}{2M(0)} \leq \frac{C_x S \rho(t) V(t)}{2M(t)} \leq \frac{C_x^0 S(\rho + \Delta\rho)(V_{cir} + \Delta V)}{2[M(0) - \Delta M]}$$

i.e.,

$$\frac{C_x \rho V S}{2M} \cong \frac{C_x^0 \rho(R_{av}) V_{cir}(R_{av}) S}{2 M_{av}} \qquad \text{etc.}$$

where

$$M_{av} = M(0) - \frac{\Delta M}{2}$$

The cited problem may be used as a model for the analysis of more complex cases.

Using the foregoing assumptions, let us write the equations of the orbital motion of the variable-mass point with aerodynamic forces:

$$\dot{u} = \frac{A_{u_1} \cos \varphi}{m} - \nu^2 x - (B_x u + B_y v) \qquad (2)$$

$$\dot{v} = \frac{A_{u_1} \sin \varphi}{m} - \nu^2 y - (B_x v - B_y u) \qquad (3)$$

$$\dot{x} = u, \qquad \dot{y} = v, \qquad \dot{m} = -\alpha u_1 \qquad (4)$$

Here u, v, x, y are projections of the velocity vector and Cartesian coordinates of the point, $m = M(t)/M(0)$ — nondimensional mass, $u_1 = P(t)/P_{max}$ — nondimensional thrust, φ angle between the thrust vector and the $0X$ — axis,

$$A = \frac{P_{max}}{M(0)} = \alpha C$$
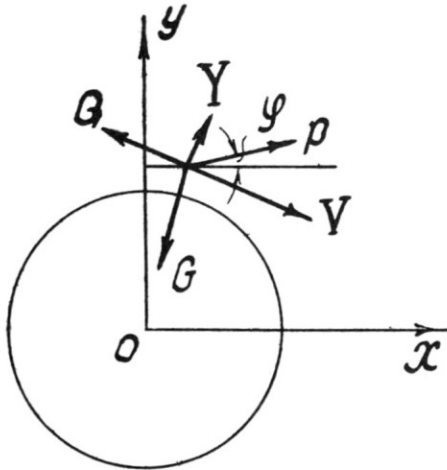
where $C = \text{const.}$ — exhaust velocity.

Figure 1.

Let us consider Meyer's problem of determining the maximum of the linear combinations $S = \sum_{i=1}^{s} C_i x_i(T)$, $C_i = $ const., composed of the finite values of the phase vector components $x = (u,v,x,y,m) = (x_1, \ldots, x_5)$, with the constraint

$$0 \leq u_1 \leq 1 \tag{5}$$

imposed on the control.

In accordance with the required maximum principle conditions [1] the optimum control providing maximum to the functional S is (see, for example, Refs. 2 and 3):

$$u_1 = Sg\vartheta = \begin{cases} 1 \text{ for } \vartheta > 0 \\ 0 \text{ for } \vartheta < 0 \end{cases} \tag{6}$$

$$\sin \varphi = -\frac{P_v}{\rho}, \quad \cos \varphi = -\frac{P_u}{\rho} \tag{7}$$

where

$$\vartheta = \rho + \frac{mPm}{C}, \quad \rho = \sqrt{P_u^2 + P_v^2}$$

The variables P_i $(i = u, \ldots, m)$ conjugated by the phase coordinate are determined by the equations:

$$\dot{P}_u = -P_z + (B_z P_u - B_y P_v) \tag{8}$$

$$\dot{P}_v = -P_y + (B_z P_v + B_y P_u) \tag{9}$$

$$\dot{P}_z = v^2 P_u, \quad \dot{P}_y = v^2 P_v, \quad P_m = -\frac{A_{u_1}}{m^2} \rho \tag{10}$$

In order to use the relations (5) and (6) and to define the form of the optimum control we shall find the linear part (8)–(10) of the conjugate system.

As noted by V. V. Sonin, Eqs. (8)–(10) can be written in a more convenient form:

$$\ddot{P}_u = B_z \dot{P}_u + v^2 P_u + B_y \dot{P}_v = 0 \tag{11}$$

$$\ddot{P}_v = B_y \dot{P}_u - B_z \dot{P}_v + v^2 P_v = 0 \tag{12}$$

As a result the corresponding characteristic equation becomes:

$$(\lambda^2 - B_x\lambda + \nu^2) + (\lambda B_y)^2 = 0 \tag{13}$$

or

$$(\lambda^2 - B_x\lambda + \nu^2 + i\lambda B_y)(\lambda^2 - B_x\lambda + \nu^2 - i\lambda B_y) = 0 \tag{14}$$

The characteristic equation (14) has four various complex roots:

$$\lambda_{1-1} = \frac{B_x}{2} \pm \sqrt{\frac{1}{2}\left[\sqrt{\left(\nu^2 - \frac{B_x^2 \pm B_y^2}{4}\right)^2 + \left(\frac{B_xB_y}{2}\right)^2} - \nu^2 + \frac{B_x^2 \pm B_y^2}{4}\right]}$$

$$\pm i\frac{B_y}{2} \pm i\sqrt{\frac{1}{2}\left[\sqrt{\left(\nu^2 - \frac{B_x^2 \pm B_y^2}{4}\right)^2 + \left(\frac{B_xB_y}{2}\right)^2} + \nu^2 - \frac{B_x \pm B_y^2}{4}\right]} \tag{15}$$

According to Eqs. (7) and (15) the optimum program of the jet force orientation is:

$$\varphi = \arctan \frac{\displaystyle\sum_{i=1}^{4} C_{v_i}e^{\lambda_i t}}{\displaystyle\sum_{i=1}^{4} C_{u_i}e^{\lambda_i t}} \tag{16}$$

where C_{u_i}, C_{v_i} ($i = 1, \ldots, 4$) are constants of integration (of which only four constants are independent) determined by the solution of the boundary-value problem.

To simplify the following analysis we shall note that the values of aero-dynamic acceleration components

$$n_x = \frac{O}{G}, \qquad n_y = \frac{Y}{G}$$

where $G = M(0) \times m(t) \times g(R_{av})$ is the instantaneous weight of the point having the mass $M(t)$ at the distance $r = R_{av}$ from the centre of gravity) are at least quantities of the first order of smallness. This follows from the requirement that the assumption (1) on an average is to be held everywhere along the trajectory including the coasting arcs. The duration of the latter may be a quantity of the same order as the period of revolution of the satellite along the circular orbit of the radius $r = R_{av}$.

Taking into account the last statement we shall give some simplified formulas for calculating the roots with the accuracy to the fifth order of smallness:

$$\lambda_{1-4} \cong \left[n_x \pm \frac{n_x n_y \sqrt{1 + (n_x^2 \pm n_y^2)}}{2} \right.$$
$$\left. \mp i n_y \pm i \sqrt{1 + (n_x^2 \pm n_y^2) + \frac{(n_x n_y)^2}{4}} \right] \nu \tag{17}$$

and to the second order of smallness:

$$\lambda_{1-4} \cong (n_x \mp i n_y \pm i) \nu \tag{18}$$

As far as the relations (18) are concerned it should be noted that having discarded the higher order smalls we have obtained multiple roots of the characteristic equation (instead of the pairs of closely related roots as it occurs in the general case).

The last note indicates that the problem is considered near the boundary of the change of the type of differential equations solution (8)–(10) (the possibility of secular terms appearance).

Therefore in this case it is necessary to use various approximate presentations of the roots with care.

The last feature is connected with the fact that the lift term is included in the equation of motion (3); in the alternate case (for $B_y = 0$) the system of equations (8)–(10) separates into two independent linear subsystems with conjugate complex roots [i.e., for $B_y = 0$ the secular terms are absent a priori in the conjugate system solution (8)–(10)].

The P — trajectory form $P_v = P_v(P_u)$ and therefore the type of solution of the system (8)–(10) make a considerable effect on the optimum programs of controlling the jet force magnitude and direction.

Let us present Eq. (15) in the form

$$\lambda_j = \mu_j + i\nu_j, \qquad (j = 1, \ldots, 4) \tag{19}$$

where because of the closely related complex roots

$$\mu_j = [n_x \pm \epsilon_1(1 \pm \epsilon_2)] \nu; \qquad \nu_j = [\tilde{\nu} \pm \delta_1(1 \pm \delta_2)] \nu;$$
$$(j = 1, \ldots, 4). \tag{20}$$

Here ϵ_1, ϵ_2 and δ_1, δ_2 are the smalls of the higher order in comparison with n_x and $\tilde{\nu}$ respectively.

Neglecting the smalls of the second order in Eq. (20) and accounting for the comments about the type of the solution of (8)–(10) we obtain finally:

$$P_i = A_i e^{(n_z + \epsilon_1)\nu t} a_i(t) \sin [\nu_1 + \Psi_i + \phi_i(t)], \qquad (i = u, v, x, y) \qquad (21)$$

where

$$a_i^2(t) = 1 + 2e^{-2\epsilon_1 \nu t} tg\omega_i \cos (2\delta_1 t + \Delta\Psi_i) + e^{-4\epsilon_1 \nu t} tg^2\omega$$

$$\sin \phi(t) = - \frac{e^{-2\epsilon_1 \nu t} tg\omega_i \sin (2\delta_1 t + \Delta\Psi_i)}{a(t)}$$

$$\cos \phi(t) = \frac{1 + e^{-2\epsilon_1 \nu t} tg\omega_i \cos (2\delta_1 t + \Delta\Psi_i)}{a(t)}$$

Here A_i, ω_i, Ψ_i, $\Delta\Psi_i$ ($i = u,v,x,y$) are *constants of integration* (of which only four constants are independent).

According to Eq. (21) in a general case P — trajectory presents a slowly spiralling out path which changes into closed curve for $B_x = B_y = 0$ (the ellipse, circle or portions of two fitting lines). These closed curves present P — trajectories of motion in the uniform central gravitational field without aerodynamic forces [2]. The pointed curves are the limited curves from which (in accordance with the relation (21) the P — trajectory spirals out (with possible self-intersections) in the case of motion with aerodynamic forces present. The degree of spiralling out [according to Eq. (21)] is determined by the aerodynamic drag value n_x while the lift value n_y defines the change of the "frequency" and the "phase" shift of the quasi-periodic function $\rho(t)$.

Figure 2 shows the typical trend of the function ρ, z and the corresponding pattern of the optimum control of the thrust magnitude $u_i(t)$. Because of the small n_x and n_y values the aerodynamic forces effects along the initial part of the trajectory are not sensible and the features of the optimum program of controlling $u_i(t)$ are the same as for the motion in vacuum [2]. Later on the aerodynamic forces effects accumulate and cause the breakdown of the former pattern of optimum control and the appearance of the new qualitative features. The main feature is that the thrust arc duration does not vanish with $t \to \infty$ as for the motion in vacuum, but on the contrary it may begin to increase starting from some revolution. For the small values of the parameter $A = P_{max}/M(0)$ the coasting arc duration may become 0 in the limit and the thrust cutoff u_i might not occur in future (i.e., the path would terminate by the thrust arc of any high duration).
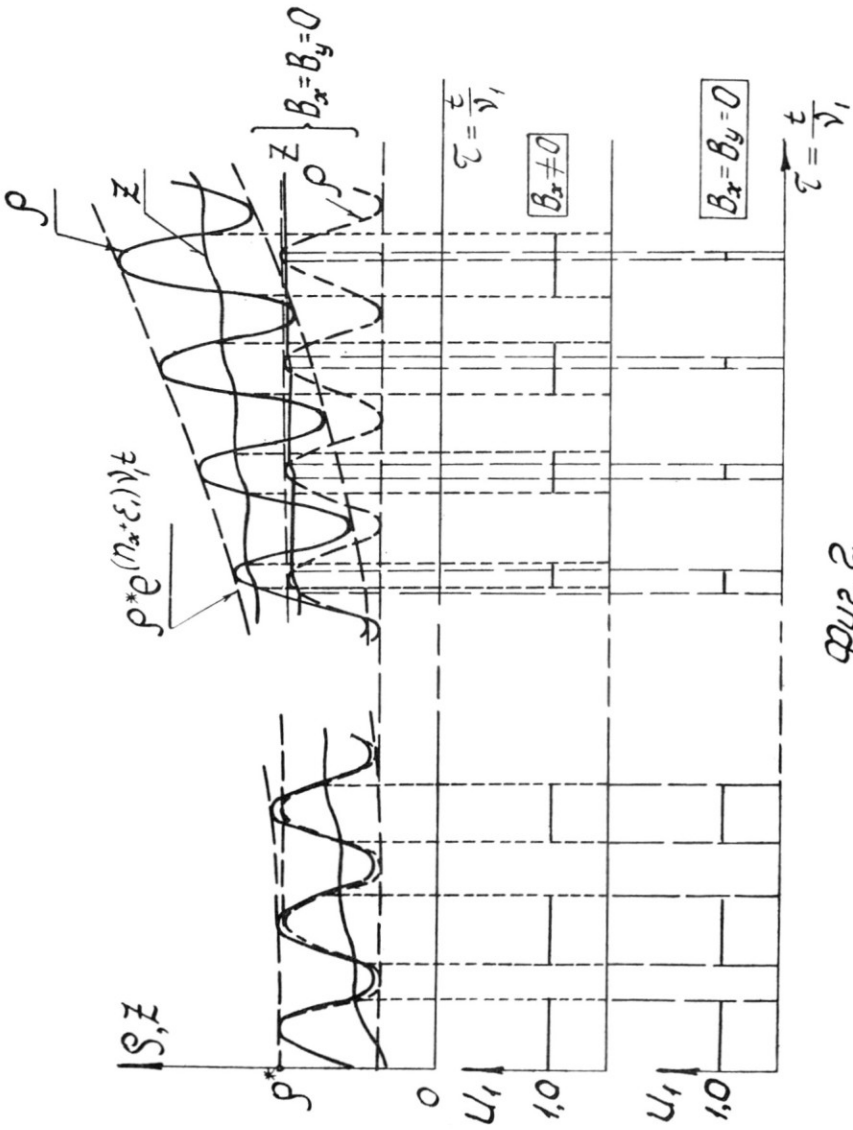
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Figure 2.

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